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THE FEFFERMAN-GRAHAM AMBIGUITY AND AdS BLACK HOLES

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Abstract

Asymptotically anti-de Sitter space-times in pure gravity with negative cosmological constant are described, in all space-time dimensions greater than two, by classical degrees of freedom on the conformal boundary at space-like infinity. Their effective boundary action has a conformal anomaly for even dimensions and is conformally invariant for odd ones. These degrees of freedom are encoded in traceless tensor fields in the Fefferman-Graham asymptotic metric for any choice of conformally flat boundary and generate all Schwarzschild and Kerr black holes in anti-de Sitter space-time. We argue that these fields describe components of an energy-momentum tensor of a boundary theory and show explicitly how this is realized in 2+1 dimensions. There, the Fefferman-Graham fields reduce to the generators of the Virasoro algebra and give the mass and the angular momentum of the BTZ black holes. Their local expression is the Liouville field in a general curved background.

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The conjectured equivalence, in the string theory approach to quantum gravity, between supergravity in D-dimensional anti-de Sitter space-time AdS_D and some quantum conformal field theory living on its boundary has drawn considerable interest [1, 2, 3]. It has been viewed as a manifestation of the holographic principle for quantum degrees of freedom [4, 5, 6]. In this paper, we show that there are *classical* degrees of freedom on the conformal boundary of AdS and we discuss their relation with the Fefferman-Graham description of gravity with negative cosmological constant. These degrees of freedom generate all AdS Schwarzschild and Kerr black holes. Their classical effective boundary action is conformally invariant for odd dimensional space-time boundaries and presents the well known conformal anomaly of gravity for even ones [7]. We derive here this anomaly for general asymptotic anti-de Sitter space-times from local transformations of the boundary, avoiding ambiguities arising from global transformations. These considerations are applied to the particular case of AdS_3 where these degrees of freedom are described locally by a Liouville field theory [8] on an arbitrary curved boundary [9, 10, 11].

In order to motivate our definition of asymptotic anti-de Sitter spaces, we first review the anti-de Sitter geometry. The anti-de Sitter space-time AdS_D in $D = d + 1$ dimensions is the hyperboloid of radius l

$$X_1^2 + ... + X_{d-1}^2 + X_d^2 - X_0^2 - X_{-1}^2 = -l^2 \tag{1}$$

embedded in the $d + 2$ flat space with metric

$$\begin{aligned} ds^2 = dX_1^2 + ... + dX_{d-1}^2 - dX_0^2 - dUdV \,, \\ X_{-1} + X_d = U \,, \\ X_{-1} - X_d = V \,. \end{aligned} \tag{2}$$

Its isometry group is $O\,(2, d)$. In terms of the coordinates $x_i = X_i/U$ and $U, U \neq 0$, the induced metric on the hyperboloid is

$$ds^2 = U^2(dx_1^2 + ... + dx_{d-1}^2 - dx_0^2) + l^2 \frac{dU^2}{U^2} \,. \tag{3}$$

One may take as boundary of the hyperboloid, the asymptotic projective cone described by Eq.(1) with $l = 0$. It can be parameterized, for $U \neq 0$, by

$$ds_\infty^2 = U^2 ds_B^2 \,, \tag{4}$$

$$ds_B^2 = (dx_1^2 + ... + dx_{d-1}^2 - dx_0^2) \,, \tag{5}$$

and we shall refer to the geometry defined by ds_B^2 as a boundary geometry. It is straightforward to check that the isometries $O\,(2,d)$ leaving the metric Eq.(4) invariant are realized in the boundary Minkowskian space-time Eq.(5) (compactified by the addition of the point at infinity corresponding to $U = 0$) by the conformal group in d dimensions. Rotations $O\,(1,d-1)$ leave U invariant while the dilations, the $2d$ translations and special conformal transformations require a compensating transformation on U. For $d=2$ the invariance of ds_∞^2 under conformal transformations of the boundary space-time is larger than $O\,(2,2)$ and is generated by two copies of the Virasoro algebra.

Spaces which, at spacelike infinity, can be described locally by the metric Eq.(4) will be called asymptotic AdS space-times. This definition has however to be made precise by specifying the limiting procedure. To this effect, it is useful to enlarge the set of boundary geometries ds_B^2 to the class of conformally flat spaces. For all $d \geq 2$, the metric Eq.(4) is then invariant for the extended conformal group $\mathcal{C}$ which leaves invariant the conformal space of boundary metrics; $\mathcal{C}$ contains as subgroups the d-dimensional reparametrization group and the Weyl group [12]. The group $\mathcal{C}$ can be generated by bulk diffeomorphisms in the following way. By writing $U^2 = y^{-1}$, the metric Eq.(3) takes the Fefferman-Graham form [13]

$$ds^2 = g_{yy}dy^2 + g_{ij}^{(d)}dx^i dx^j = \frac{l^2 dy^2}{4y^2} + \frac{1}{y}\tilde{g}_{ij}(x,y)dx^i dx^j \tag{6}$$

where the d-dimensional metric $\tilde{g}_{ij}(x,y)$ is simply the y-independent Minkowskian boundary metric Eq.(5). We now take $\tilde{g}_{ij}(x,y)$ to be any metric which tends, when $y \to 0$, to a conformally flat metric $\tilde{g}_{(0)ij}(x)$. The diffeomorphisms keeping the form Eq.(6) are those for which the Lie derivative of g_{yy} and g_{yi} vanish. They are given by [14]:

$$\begin{aligned}\delta y &= 2\sigma(x)y\,, & (7)\\ \delta x^i &= -\frac{l^2}{2}\int_0^y \tilde{g}^{ij}(x,y')\sigma(x)_{,j}dy' + \phi^i(x)\,, \quad \phi^i(x)\,\text{arbitrary}\,, & (8)\end{aligned}$$

and induce on $\tilde{g}_{(0)ij}(x)$, up to a reparametrization engendered by $\phi_i(x)$, the Lie variation:

$$\delta\tilde{g}_{(0)ij}(x) = -2\sigma(x)\tilde{g}_{(0)ij}(x)\,. \tag{9}$$

Thus these diffeomorphisms form the group $\mathcal{C}$ acting on the AdS boundary metrics. Compensating the transformation Eq.(9) by a Weyl transformation,

we recover the invariance of the metric Eq.(4) for the extended conformal group $\mathcal{C}$. Note that this result follows from geometry and does not require any gravitational field equations.

Given the conformally flat metric $\tilde{g}_{(0)ij}(x)$, it is possible to reconstruct the bulk metrics of the type Eq.(6) which satisfy in a neighborhood $\mathcal{V}$ of $y = 0$ Einstein equations for pure gravity with negative cosmological constant $\Lambda \equiv -d(d-1)/l^2$. The reconstruction is however not unique and yields, in addition to AdS, distinct asymptotic anti-de Sitter space-times.

In the metric Eq.(6), the Einstein equations [7, 15] take the form

$$l^2 \tilde{R}^i_j + (d-2)h^i_j + h^k_k \delta^i_j - y(2\partial_y h^i_j + h^k_k h^i_j) = 0 \tag{10}$$

where

$$h^i_j = \tilde{g}^{ik} \partial_y \tilde{g}_{kj} \,, \tag{11}$$

together with the y-lapse constraint

$$\begin{aligned} R^y{}_y - \frac{1}{2}R - d(d-1)/2l^2 &= 0 \\ \textit{or} \qquad l^2\tilde{R} + 2(d-1)h^i_i + y(h^i_j h^j_i - h^i_i h^j_j) &= 0 \,, \end{aligned} \tag{12}$$

and the d y-shift constraints

$$\begin{aligned} R^y{}_j &= 0 \\ \textit{or} \qquad h^i_{j;i} - h^k_{k,j} &= 0 \,. \end{aligned} \tag{13}$$

Eq.(10) can be solved iteratively by expanding $\tilde{g}_{ij}(x, y)$ in a power series expansion in y. The generic term in the expansion is $\tilde{g}_{(2k)}(x)\, y^k$ up to $k = d/2 - 1$ for d even and to $(d-1)/2$ for d odd. The coefficients of this expansion are obtained algebraically in terms of $\tilde{g}_{(0)}(x)$ up to $\tilde{g}_{(d-2)}(x)$ for d even and $\tilde{g}_{(d-1)}(x)$ for d odd. For d even, there may be, in addition of $\tilde{g}_{(d)}(x)\, y^{d/2}$ a term $\tilde{k}_{(d)}(x)\, y^{d/2} \ln y$ where $\tilde{k}_{(d)}$ is traceless, namely $Tr\, \tilde{g}^{-1}_{(0)} \tilde{k}_{(d)} = 0$. While $\tilde{k}_{(d)}(x)$ is still determined by $\tilde{g}_{(0)}(x)$, the traceless part $\tilde{g}^t_{(d)}(x)$ of $\tilde{g}_{(d)}(x)$ is algebraically undetermined. For d odd, there is also a term $\tilde{g}^t_{(d)}(x)\, y^{d/2}$, where $\tilde{g}^t_{(d)}(x)$ is traceless and is algebraically undetermined [13, 16]. We shall refer to these undeterminacies as the Fefferman-Graham ambiguity. All higher order terms in the expansion can be expressed algebraically in terms of $\tilde{g}_{(0)}(x)$ and $\tilde{g}^t_{(d)}(x)$. To determine a solution in $\mathcal{V}$, one must specify the transverse traceless fields $\tilde{g}^t_{(d)}(x)$: although algebraically undetermined, these fields must

satisfy the differential equations obtained by expanding in y the y-shift constraint Eq.(13). Note that Eq.(12) does not introduce new information.

We define as asymptotic anti-de Sitter space-time $\mathcal{G}$ any space-time which can in $\mathcal{V}$ be parameterized by a Fefferman-Graham metric Eq.(6) such that:
(i) $\tilde{g}_{(0)ij}(x) = \lim_{y\to 0} \tilde{g}_{ij}(x,y)$ is conformally flat,
(ii) the following boundary fields:

$$\tilde{g}_{(2)}(x), \tilde{g}_{(4)}(x) \quad ....\tilde{g}_{(d-1)}(x) \quad d\ odd\,, \tag{14}$$

$$\tilde{g}_{(2)}(x), \tilde{g}_{(4)}(x) \quad ....\tilde{g}_{(d-2)}(x), Tr\,\tilde{g}_{(d)}(x), \tilde{k}_{(d)}(x) \quad d\ even\,, \tag{15}$$

are expressed algebraically in terms of $\tilde{g}_{(0)}(x)$ in accordance with Eq.(10)[1]. While Eqs.(14) and (15) constitute boundary conditions, the algebraically undetermined fields $\tilde{g}^t_{(d)}(x)$ encode boundary dynamical degrees of freedom whose nature will be examined later. We first show that there exists a finite action functional for these degrees of freedom and derive its properties under the conformal group $\mathcal{C}$.

The Einstein-Hilbert action with suitable boundary terms is, in $\mathcal{V}$,

$$S = S_0 + \frac{1}{16\pi G}\int_{\mathcal{V}} \sqrt{|g|}(R + \frac{d(d-1)}{l^2})d^{d+1}x + \frac{1}{8\pi G} I\ , \tag{16}$$

$$I \equiv -\int_{\mathcal{S}} \sqrt{|g^{(d)}|}(K - C)d^d x\ . \tag{17}$$

where $\mathcal{S}$ is a $y = \bar{y}$, d-dimensional, boundary in $\mathcal{V}$ with topology $S_{d-1} \times R$ and S_0 is the contribution to S outside $\mathcal{V}$. The surface term I is introduced, as usual, to render the action stationary for solutions of Einstein's equations with fixed fields on the boundary $\mathcal{S}$ [17]. $K = K^i_i$ where K_{ij} is the extrinsic curvature tensor in the metric $g^{(d)}_{ij}$ of Eq.(6). The constant C is introduced for convenience and will be fixed later.

We perform a diffeomorphism in D dimensions defined by an infinitesimal displacement field $\xi^\mu(x,y)$ which vanishes outside $\mathcal{V}$. To compute the corresponding variation of the action Eq.(16) we first write the surface integral in covariant form by introducing the normal D-vectors n_μ to the surface elements $d\Sigma_\mu$ on $\mathcal{S}$. By embedding the displaced surface $\mathcal{S}$ under the diffeomorphism in a family of surfaces characterized by infinitesimal displacements $\alpha\,\xi^\mu$ where α varies from 0 to 1, one defines a field n_μ normal to the surfaces α = constant. The extrinsic curvature scalar K can be written as a

[1] The work of reference [14] suggests that these boundary conditions are in fact a direct consequence of the transformations Eqs.(7) and (8).

D-dimensional scalar $K \equiv -n^\mu{}_{;\mu}$. Using

$$\sqrt{|g^{(d)}|}d^d x = \sqrt{|g|}n^\mu d\Sigma_\mu \, , \tag{18}$$

the integral Eq.(17) becomes

$$I = \int_{\mathcal{S}} \sqrt{|g|}n^\mu (n^\sigma_{\ \sigma} + C) d\Sigma_\mu \; . \tag{19}$$

The variation of the action is obtained by taking the Lie derivative of the integrand of volume terms after transforming the surface term to a volume integral of a D-divergence. We get

$$\delta_\xi S = \frac{1}{16\pi G}\int_{\mathcal{S}} \sqrt{|g|}\{(R + \frac{d(d-1)}{l^2}) + 2[n^\mu (n^\sigma{}_{;\sigma} + C)]_{;\mu}\}\xi^\rho d\Sigma_\rho \, . \tag{20}$$

We substitute in Eq.(20) the identity

$$(n^\mu n^\sigma{}_{;\sigma})_{;\mu} = n^\mu{}_{;\mu} n^\sigma{}_{;\sigma} - n^\mu{}_{;\sigma} n^\sigma{}_{;\mu} - n^\mu n^\sigma R_{\sigma\mu} + (n^\mu n^\sigma{}_{;\mu})_{;\sigma} \, . \tag{21}$$

It is easily checked that on $\mathcal{S}$ the last term in Eq.(21) gives no contribution. If we insert the y-lapse constraint Eq.(12) in the Gauss-Codazzi equation

$$2G^y{}_y \equiv 2R^y{}_y - R = -R^{(d)} + [K^i_i K^j_j - K^i_j K^j_i] \tag{22}$$

and use the identity $\sqrt{|g|}\xi^\rho d\Sigma_\rho = \sqrt{|g^{(d)}|}\xi d^d x$ with $\xi = \xi^\mu n_\mu$, we get

$$\delta_\xi S = \frac{1}{8\pi G}\int_{\mathcal{S}} \sqrt{|g^{(d)}|}\{R^{(d)} + \frac{d(d-1)}{l^2} + 2CK\}\xi d^d x \, . \tag{23}$$

Taking $C = (d-1)/l$, this can be written as

$$\delta_\xi S = \frac{l}{16\pi G}\int_{\mathcal{S}} \sqrt{|\tilde{g}^{(d)}|}(R^{(d)} + \frac{d-1}{l^2} h^i_{\ i}\,)/\bar{y}^{d/2}\, \delta\bar{y}\, d^d x \, . \tag{24}$$

Eq.(24) is similar to the equation obtained in reference [15] for the variation of the Einstein-Hilbert action but there are two noticeable differences. First, our result is valid for arbitrary *local* variations $\delta\bar{y}$ around the surface $y = \bar{y}$. Second, as shown below, in the limit $\bar{y} \to 0$ Eq.(24) does not require the evaluation of the action Eq.(16) on a solution of Einstein's equations but only on an arbitrary asymptotically anti-de Sitter space-time $\mathcal{G}$.

We examine Eq.(24) when $\bar{y}$ tends to zero. The integral is divergent and we may classify the divergences by expanding the integrand in power series in $\bar{y}$. We expand the numerator of the integrand of Eq.(24), keeping only non vanishing terms as $\bar{y}$ goes to 0. These divergences arise from terms of order less than $\bar{y}^{d/2}$ in the expansion of $\sqrt{|\tilde{g}^{(d)}|}$, $R^{(d)}$ and h^i_i, and potentially from terms in h^i_i (which contains a $\bar{y}$-derivative) of order $\bar{y}^{d/2}$ or $\bar{y}^{d/2}\ln\bar{y}$ in the expansion of the metric $\tilde{g}_{ij}(x,y)$. However, due to tracelessness, there is no contribution containing $\tilde{g}^t_{(d)}$. (For the same reasons, there is no contribution from the logarithmic term for d even.) The divergent contributions depend only on the fields appearing in Eqs.(14) and (15) for which the y-constraint used in the derivation of Eq.(24) reduces to an identity. Thus, as announced, in the limit $\bar{y}\to 0$, this equation is valid for all asymptotic anti-de Sitter space-times $\mathcal{G}$. The divergent terms have no dynamical content.

We denote by $S(\mathcal{G})$ the action S evaluated on a space $\mathcal{G}$. Integrating Eq.(24) for a global variation $\delta\bar{y}$ one gets when $\bar{y}\to 0$

$$S(\mathcal{G}) = \frac{A_{(d/2)-1}}{\bar{y}^{(d/2)-1}} + ... + \frac{A_{1/2}}{\bar{y}^{1/2}} + S(\mathcal{G})_{fin}\,, \qquad d\ odd \tag{25}$$

$$S(\mathcal{G}) = \frac{A_{(d/2)-1}}{\bar{y}^{(d/2)-1}} + .. + A_0\ln\bar{y} + S(\mathcal{G})_{fin}\,, \qquad d\ even \tag{26}$$

where $S(\mathcal{G})_{fin}$ is independent of $\bar{y}$ and finite. The fields $\tilde{g}^t_{(d)}(x)$ enter only $S_{fin}(\mathcal{G})$ which contains therefore all the dynamics. One may view $S_{fin}(\mathcal{G})$ as an effective boundary action for these dynamical degrees of freedom.

Comparing Eqs.(25) and (26) with Eq.(24) for a local variation of $\delta\bar{y}$, we see that the coefficients of the divergent terms A can be expressed as surface integral of local functions of $\tilde{g}_{(0)}(x)$ only. In particular we may write

$$A_0 = \frac{1}{2}\int\sqrt{|g^{(d)}|}\mathcal{A}(\tilde{g}_{(0)}(x))d^dx, \tag{27}$$

where $\sqrt{|g^{(d)}|}\mathcal{A}/2$ is the coefficient of $\delta\bar{y}/\bar{y}$ in Eq.(24).

Taking for $\delta\bar{y}$ the particular local transformation Eq.(7) renders the contribution of the $\mathcal{A}$ term in Eq.(24) finite. The unsubtracted action $S(\mathcal{G})$ is invariant under the combined Lie transformation Eq.(24) and (minus) the conformal transformation of Eq.(9) acting on $\tilde{g}_{(0)}(x)$, provided the conformal transformation of the field $\tilde{g}^t_{(d)}(x)$ is defined by (minus) the Lie variation under the diffeomorphism Eqs.(7) and (8). This expresses the invariance of ds^2_∞ under the group $\mathcal{C}$. The local divergent integrals are separately invariant

under these transformations except for the logarithmic divergent integral A_0 which is conformally invariant. The compensation of the finite $\delta\bar{y}/\bar{y}$ term identifies $\mathcal{A}(\tilde{g}_{(0)}(x))$ to the well known anomaly of the "renormalized" action $S_{fin}(\mathcal{G})$ for d even while for d odd $S_{fin}(\mathcal{G})$ is conformally invariant [3, 7]. As our proof uses *local* variations of $\bar{y}$, it is free of ambiguities hidden in proofs based on global variations and its validity extends to all asymptotic anti-de Sitter space-times $\mathcal{G}$.

We now discuss the nature of the boundary degrees of freedom encoded in $\tilde{g}^t_{(d)}(x)$.

When $\mathcal{G}$ solves the Einstein's equations, $\tilde{g}_{(0)}(x)$ can still be taken as an arbitrary conformally flat metric but $\tilde{g}^t_{(d)}(x)$ must satisfy all Einstein's equations, which means that it must solve the y-shift constraints Eq.(13). Expanding $h^i_{\ j;i} - h^k_{\ k,j}$ in a power series in y, all terms of order less than $y^{d/2-1}$ vanish identically because they contain only coefficients $\tilde{g}_{(2k)}(x)$ which are already determined algebraically in terms of $\tilde{g}_{(0)}(x)$. The term in $y^{d/2-1}$ yields a differential equation for $\tilde{g}^t_{(d)}(x)$. From Eq.(13), we can see that this equation takes the form

$$D_i\, \tilde{g}_{(d)j}^{\ i} + \psi_j[\tilde{g}_{(0)}(x)] = 0 \tag{28}$$

where here and in what follows, indices are raised with the boundary metric $\tilde{g}_{(0)}(x)$ and D_i is the covariant derivative in the same metric. The quantities $\psi_i[\tilde{g}_{(0)}(x)]$ contain only terms determined in terms of derivatives of $\tilde{g}_{(0)}(x)$, hence expressible in terms of curvature terms of the boundary metric, and depend explicitly on d. Note that Eq.(28), which gives equations of motion for the degrees of freedom on the boundary of AdS_{d+1}, reduces to an identity for all higher dimensional boundaries, when the explicit dependence of $\psi_i[\tilde{g}_{(0)}(x)]$ on the dimension is taken into account.

We illustrate Eq.(28) for the cases $d = 2, 3$ and 4. When $d = 2$ the order zero in y in Eq.(13) yields

$$D_i(\tilde{g}_{(2)j}^{\ i} + \delta^i_{\ j}\frac{l^2}{2}\tilde{R}) = 0\,, \tag{29}$$

where we have used Eq.(10) to get the trace part of $\tilde{g}_{(2)}$ in terms of $\tilde{g}_{(0)}(x)$:

$$\tilde{g}_{(2)i}^{\ i} = -\frac{l^2}{2}\tilde{R}\,. \tag{30}$$

For $d = 3$ we take the derivative of Eq.(13) with respect to $y^{1/2}$; the Christoffel symbols are not affected at this order and taking into account the vanish-

ing of the trace $g^i_{(3)i}$, we simply get

$$D_i\,\tilde g_{(3)j}^{\ i} = 0\,. \tag{31}$$

For $d = 4$, we take the derivative with respect to y and take into account the change of the Christoffel symbols in the covariant derivative. We get

$$D_i\,\{\tilde g_{(4)j}^{\ i} + \frac{l^4}{16}[\tilde R \tilde R^i_j - 2\tilde R^{im}\tilde R_{mj} + \frac{1}{2}\delta^i_j(\tilde R^{mn}\tilde R_{mn} - \frac{5}{9}\tilde R^2)]\} = 0\,. \tag{32}$$

The form of Eq.(29), (31) and (32) can be summarized by

$$D_i(\tilde g_{(d)j}^{\ i} + \xi^i_{(d)j}) = 0 \tag{33}$$

where $\xi_{(d)j}^{\ i}$ is constructed out of the curvature tensors and their derivatives. From the conservation law Eq.(33) we may define a conserved tensor

$$T^i_{(d)j} = \frac{1}{\alpha_d}(\tilde g_{(d)j}^{\ i} + \xi_{(d)j}^{\ i}) \tag{34}$$

where α_d is a numerical coefficient. Choosing $\alpha_2 = 8\pi G l$ and $\alpha_4 = 4\pi G l$, we get

$$\begin{aligned} T^i_{(2)i} &= \frac{3l}{2G}\frac{\tilde R}{24\pi}, && (35)\\ T^i_{(3)i} &= 0, && (36)\\ T^i_{(4)i} &= \frac{l^3}{8\pi G}(\frac{1}{8}\tilde R^{mn}\tilde R_{mn} - \frac{1}{24}\tilde R^2). && (37) \end{aligned}$$

We see that the traces of the covariantly conserved tensors $T^i_{(d)j}$, which are related by Eq.(34) to the Fefferman-Graham fields $\tilde g^t_{(d)}(x)$, reproduce for $d = 2$ and 4 the gravitational anomalies $\mathcal{A}$ computed from Eq.(24) and its vanishing for $d = 3$. Note that in flat space the equations of motion Eq.(28) express already for all d the conservation of the quantity $\tilde g_{(d)j}^{\ i}$, because then $\psi_i[\tilde g_{(0)}(x)]$ vanishes. The extension of this conservation equation to the case of a general boundary metric $\tilde g_{(0)}(x)$ relies on the precise form of $\psi_i[\tilde g_{(0)}(x)]$ and does not involve the dynamical fields $\tilde g^t_{(d)}(x)$. These results suggests that, for all d, the degrees of freedom hidden in the Fefferman-Graham ambiguity $\tilde g^t_{(d)}(x)$ can be expressed in terms of a conserved energy momentum tensor of some boundary fields. The trace of this energy-momentum tensor would, on the

equations of motion, be equal to the trace anomaly $\mathcal{A}$. These boundary fields would describe the shape of the surface $\mathcal{S}$ in the limiting process $\bar{y} \to 0$.

The boundary degrees of freedom generate all Schwarzschild and Kerr black holes in asymptotic AdS_D space-time for all D. The metric of AdS Schwarzschild black holes of mass M can be taken to be

$$ds^2 = -(1 - \frac{\lambda}{r^{d-2}} + \frac{r^2}{l^2})dt^2 + (1 - \frac{\lambda}{r^{d-2}} + \frac{r^2}{l^2})^{-1}dr^2 + r^2 d\Omega_{d-1}^2 \,, \tag{38}$$

where $\lambda = \nu_D G_D M$. G_D is the gravitational constant in $D = d+1$ dimensions and ν_D is a D-dependent numerical coefficient. When $r \to \infty$, the relation between r and the variable y in the metric Eq.(6) is $dr/r \to -dy/2y$ or equivalently by suitable choice of the integration constant constant, $r/l \to y^{-1/2}$. Hence the leading order in y of the mass term in the coefficient of dt^2 is $O(d/2)$. As the mass term has no counterpart in the pure AdS_{d+1} geometry its leading contribution to the expansion of $\tilde{g}_{ij}(x,y)$ is the traceless quantity $\tilde{g}^t_{(d)}(x)$ which is here a constant independent of x.

The tracelessness of the mass term can be verified explicitly. The relation between r and y is defined by

$$(1 - \frac{\lambda}{r^{d-2}} + \frac{r^2}{l^2})^{-1/2} dr = -\frac{l dy}{2y} \,. \tag{39}$$

Posing, for $d > 2$, $2r/l = \xi - 1/\xi$, one obtains, up to order ξ^{-d},

$$\ln \xi - \frac{2^{d-1}}{l^{d-2}\lambda^2 d}\xi^{-d} = -\frac{1}{2}\ln\frac{y}{4} \,. \tag{40}$$

Expressing this equality to the same order in terms of r, one gets

$$\frac{r}{l} = \frac{1}{y^{1/2}}(1 + \frac{1}{4}y + \frac{\lambda^2}{2l^{d-2}d}y^{d/2}) \,, \tag{41}$$

from which one easily verifies that $\tilde{g}^{ik}_{(0)}\tilde{g}_{(d),ki}$ vanishes. For $d = 2$, the exact relation between r and y is obtained along similar lines by the change of variable $2r/l = \xi + 1/\xi$ and the vanishing of $\tilde{g}^{ik}_{(0)}\tilde{g}_{(2),ki}$ follows.

These conclusions can be extended to Kerr black holes. The dependence on r and hence of the leading order in y of the mass term in Eq.(38) follows from dimensional considerations : $G_D M$ has dimension $D-3 = d-2$ and its contribution to g_{tt} cannot depend on l. Similarly the angular momentum J

will give a term proportional to $G_D J$ in $g_{t\phi}$ of the same dimensionality. This is the leading dependence on the angular momentum in the Kerr metric and is again encoded in the traceless field $\tilde{g}^t_{(d)}(x)$.

We now apply these considerations to asymptotic AdS_3 space-times [9]. When $d = 2$, Eq.(26) gives

$$S(\mathcal{G}) = A_0 \ln \bar{y} + S_{fin}(\mathcal{G})\,. \tag{42}$$

Eq.(35) shows that the conformal anomaly, computed from Eq.(24) (see also [18])

$$\mathcal{A} = \frac{3l}{2G}\frac{\tilde{R}}{24\pi} \tag{43}$$

agrees with the trace of the conserved energy momentum tensor $T^{\ i}_{(2)j}$ defined by Eqs.(29) and (34)

$$\tilde{g}^{\ i}_{(2)j} + \delta^i_{\ j}\frac{l^2}{2}\tilde{R} = 8\pi G l T^{\ i}_{(2)j}\,. \tag{44}$$

To verify that $T^{\ i}_{(2)j}$ is indeed the energy-momentum tensor of a boundary field, we consider the Weyl transformation of $\tilde{g}_{(0)ij}(x)$ and $\tilde{g}_{(2)ij}(x)$ given by (minus) the Lie derivatives defined by the diffeomorphism Eqs.(7) and (8). We have [14]

$$\begin{aligned} \delta_{Weyl}\,\tilde{g}_{(0)ij}(x) &= 2\sigma(x)\tilde{g}_{(0)ij}(x)\,, & (45)\\ \delta_{Weyl}\,\tilde{g}_{(2)ij}(x) &= l^2(\sigma(x))_{,i\,;j}\,. & (46)\end{aligned}$$

Taking a scalar field ϕ such that e^ϕ has conformal weight -1 ($\delta_{Weyl}\,\phi = -\sigma(x)$), one obtains from Eq.(46)

$$\tilde{g}_{(2)ij} = l^2[-\phi_{,i\,;j} + \phi_{,i}\phi_{,j} + \tilde{g}_{(0)ij}(\lambda e^{2\phi} - \frac{1}{2}\tilde{g}^{kl}_{(0)}\phi_k\phi_l)]\,, \tag{47}$$

where we have included the general conformal invariant term $\lambda g_{(0)ij}e^{2\phi}$, with arbitrary λ. This equation identifies T_{ij} in Eq.(44) as the energy-momentum tensor derived from the Liouville action

$$S_{Liouville} = \frac{-l}{8\pi G}\int\sqrt{|\tilde{g}_{(0)}|}(\frac{1}{2}\tilde{g}^{ij}_{(0)}\phi_{,i}\,\phi_{,j} + \frac{1}{2}\tilde{R}\phi + \lambda e^{2\phi})d^2x\,, \tag{48}$$

computed on the Liouville equations of motion. As it can be shown that Eq.(47) is integrable on the equations of motion, we have identified locally

all the degrees of freedom encoded in the Fefferman-Graham traceless field $\tilde{g}^t_{(2)}(x)$ to the Liouville field on a arbitrary curved 2-dimensional background and the Liouville action is locally equivalent to $S_{fin}(\mathcal{G})$. The action Eq.(48) is the generalization to curved boundaries [10] of the realization given in reference [8] of the Brown-Henneaux central charge [19] encoded in Eq.(43). The traceless components in Eq.(47) give the mass and angular momentum [19] of all the BTZ black holes [20, 21]

$$\frac{8\pi G}{l}T_{++} = \frac{1}{4}(M - J/l)\,, \quad \frac{8\pi G}{l}T_{--} = \frac{1}{4}(M + J/l)\,. \tag{49}$$

To conclude, we stress that, for asymptotic anti-de Sitter space-times, the quantum holographic principle seems to have a classical ancestor. What is left of this feature in the limit $\Lambda \to 0$ is unclear to us.

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References

[1] J. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity", Adv. Theor. Math. Phys. **2** (1998) 231, Int. J. Theor. Phys. **38** (1999) 1113, hep-th/9711200.

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, "Gauge Theory Correlators from Noncritical String Theory", Phys. Lett. **B428** (1998) 105, hep-th/9802109.

[3] E. Witten, "Anti-de Sitter Space and Holography", Adv. Theor. Math. Phys. **2** (1998) 253, hep-th/9802150.

[4] G. 't Hooft, "Dimensional Reduction in Quantum Gravity", in "Salamfestschrift: a Collection of Talks", World Scientific Series in 20th Century Physics, vol. **4**, Eds. A. Ali, J. Ellis and S. Randjbar-Daemi (Worlk Scientific, 1993), gr-qc/9310026.

[5] L. Susskind, "The World as an Hologram", J. Math. Phys. **36** (1995) 6377, hep-th/9409089.

[6] L. Susskind and E. Witten, "The Holographic bound in anti-de Sitter Space", hep-th/9805114.

[7] M. Henningson and K. Skenderis, "The Holographic Weyl Anomaly", JHEP **9807** (1998) 023, hep-th/9806087.

[8] O. Coussaert, M. Henneaux and P. van Driel, "The Asymptotic Dynamics of Three-Dimensional Einstein Gravity with a Negative Cosmological Constant", Class. Quant. Grav. **12** (1995) 2961, gr-qc/9506019.

[9] K. Bautier, "Diffeomorphisms and Weyl Transformations in AdS_3 Gravity", Presented at Meeting on Quantum Aspects of Gauge Theories, Supersymmetry and Unification, Paris, France, 1-7 Sep 1999, hep-th/9910134.

[10] M. Rooman and P. Spindel, "Aspects of (2+1)-Dimensional Gravity: AdS_3 Asymptotic Dynamics in the Framework of Fefferman-Graham-Lee Theorems", Presented at International European Conference on Gravitation: Journées Relativistes 99, Weimar, Germany, 12-17 Sep 1999, hep-th/9911142.

[11] K. Skenderis and N. Solodukhin, "Quantum Effective Action from the AdS/CFT Correspondence", Phys. Lett. **B343** (2000) 316, hep-th/9910023.

[12] T. Fulton, R. Rohrlich and L. Witten, "Conformal Invariance in Physics", Rev. Mod. Phys. **34** (1960) 442.

[13] C. Fefferman and C.R. Graham, "Conformal Invariants", in "Elie Cartan et les Mathématiques d'Aujourd'hui", Astérisque (1985) 95.

[14] C. Imbimbo, A. Schwimmer, S. Theisen and S. Yankielowicz, "Diffeomorphisms and Holographic Anomalies", hep-th/9910267.

[15] W. Mück and K.S. Viswanathan, "Counterterms for the Dirichlet Prescription of the AdS/CFT Correspondence", hep-th/9905046.

[16] C.R. Graham, "Volume and Area Renormalizations for Conformally Compact Einstein Metrics", in Proceedings of the 19th Winter School in Geometry and Physics, Srni, Czech Republic, 9-16 Jan 1999, math.dg/9909042.

[17] G.W. Gibbons and S.W. Hawking, "Action Integrals and Partition Functions in Quantum Gravity", Phys. Rev. **D15** (1977) 2752.

[18] M. Nishimura and Y. Tanii, "Super Weyl Anomalies in the AdS/CFT Correspondence", Int. J. Mod. Phys. **A14** (1999) 3731, hep-th/9904010.

[19] J.D. Brown and M. Henneaux, "Central Charge in the Canonical Realization of Asymptotic Symmetries : an Example from Three-Dimensional Gravity", Commun. Math. Phys. **104** (1986) 207.

[20] M. Bañados, C. Teitelboim and J. Zanelli, "The Black Hole in Three-Dimensional Space-Time", Phys. Rev. Lett. **69** (1992) 1849, hep-th/9204099.

[21] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, "Geometry of the (2+1) Black Hole", Phys. Rev. **D48** (1993) 1506, gr-qc/9302012.